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4TH-ORDER TENSOR INVARIANTS AND GEOMETRIC REPRESENTATION

BY

EDWARD M. WU

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Arlington, Va., 22217

PROGRAM MANAGER
ROLF BUCHDAHL

MONSANTO RESEARCH CORPORATION
A SUBSIDIARY OF MONSANTO COMPANY
800 N. LINDBERGH BOULEVARD ST. LOUIS, MISSOURI 63166

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EDWARD M. WU

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FOREWORD

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4th-ORDER TENSOR INVARIANTS AND GEOMETRIC REPRESENTATION

Edward M. Wu*

ABSTRACT

First and second order invariants of 4th-order tensor are derived. Geometric representations analogous to Mohr's circle are presented to aid the visualization and operations of 4th-order tensors such as transformation, determination of principal direction and optimization procedures. Possible applications of the invariants are suggested.

*Assistant Professor, Materials Research Laboratory, Mechanical and Aerospace Engineering, Washington University, St. Louis, Missouri.

Introduction

Many physical properties of composites and crystals are functions of material orientations. Such properties can be characterized by the appropriate constitutive functions. In order to insure the invariancy with respect to coordinate transformations, these functions are customarily expressed in tensorial forms. Additional groundwork is required in practical engineering application of such constitutive functions. Material constants must be measured for design computations, and analytical techniques must be explored for operational efficiency. Both experimental techniques and operational procedures are well established for physical properties which are scalars, vectors and second order tensors. For example, the measurement of strain tensor, as well as its analysis, visualization and transformation by the Mohr's Circle are well known. However, comparable operations for 4th-order tensors have not been fully explored. In the advent of composites as anisotropic engineering materials, the analysis and operation of 4th-order tensors becomes a practical necessity. Several familiar examples of technical 4th-order tensors are: elastic compliance, electrostriction, 2nd-order term of non-linear thermal expansion and environmental swelling. There remain many other important directional dependent physical properties such as strength and certain transport phenomena whose tensorial characteristics have yet to be adequately verified experimentally. In order to facilitate the analysis, measurement and operation of such physical properties, we derive the invariants of a 4th-order tensor and suggest a geometric interpretation.

Invariants of 4th-Order Tensor

Tensor representation of physical properties which are directionally dependent for engineering application is customarily in Cartesian coordinates. For this reason the derivation of invariants of a 4th-order Cartesian tensor is discussed. To avoid distinction between pseudo-invariants and invariants, only right-handed Cartesian coordinates are used. The invariants derived are for rotation about the x_3 axis. Companion invarinats for other axes of rotation can be derived through simple permutation of the indicies. At different stages of the derivation, the symmetry conditions of the 4th-order tensor are used to simplify the algebra. Appropriate modifications which are cumbersome but straightforward must be made for skew-symmetric tensors.

We consider a 4th-order symmetric, Cartesian tensor S_{ijkl} and note that from tensor algebra that its scalar contractions are invariant to orthogonal coordinate transformations. We shall seek these scalar contractions and find the number of independent invariants.

A 4th-order tensor S_{ijkl} can be contracted through combinations of the substitution tensor δ_{ij} and the permutation tensor ε_{ijk} which are defined as:

$$\delta_{ij} = \begin{bmatrix} 100 \\ 010 \\ 001 \end{bmatrix} \tag{1}$$

and $$\varepsilon_{ijk} = \begin{cases} +\ 1 & \text{for even permutation of ijk} \\ -\ 1 & \text{for odd permutation of ijk} \\ \ \ 0 & \text{for no permutation} \end{cases}$$

The 2nd-order tensor components of the 4th-order tensor can be obtained by contraction, they are:

$$\delta_{ij}\ S_{ijkl} \equiv D_{kl} \tag{2}$$

$$\delta_{ik}\ S_{ijkl} \equiv D'_{jl} \tag{3}$$

Equations (2) and (3) are the only distinct components because of symmetry, i.e.,

$$\begin{aligned} S_{ijkl} &= S_{ijlk} \\ S_{ijkl} &= S_{klij} \end{aligned} \tag{4}$$

The 3rd-order components of the 4th-order tensor are:

$$\varepsilon_{mik}\ S_{ijkl} \equiv Q_{mjl} \tag{5}$$

$$\varepsilon_{mij}\ S_{ijkl} \equiv Q'_{mkl} \tag{6}$$

Again, from symmetry conditions Eq.(4), Eqs.(5 and 6) are the only distinct components. Furthermore, it can be readily shown from the properties of the permutation tensor ε_{ijk} that $Q'_{mkl} = 0$. The 2nd-order components of Q_{mjl} then takes the form:

$$\varepsilon_{njl}\ Q_{mjl} = \varepsilon_{njl}\ \varepsilon_{mik}\ S_{ijkl} \equiv T_{nm} \tag{7}$$

$$\varepsilon_{nkl}\ Q'_{mkl} = \varepsilon_{nkl}\ \varepsilon_{mij}\ S_{ijkl} \equiv T'_{nm} = 0 \tag{8}$$

First and Second order invariants can now be derived for those 2nd-order tensor components $\left(D_{kl},\ D'_{jl},\ T_{nm} \right)$ of the 4th-order tensor S_{ijkl}.

The first order invariants are:

$$\delta_{kl}\ D_{kl} = \delta_{kl}\ \delta_{ij}\ S_{ijkl} \equiv I_1 \tag{9}$$

$$\delta_{jl}\ D'_{jl} = \delta_{jl}\ \delta_{ik}\ S_{ijkl} \equiv I'_1 \tag{10}$$

$$\delta_{nm}\ T_{nm} = \delta_{nm}\ \varepsilon_{njl}\ \varepsilon_{mik}\ S_{ijkl} \equiv -(1/2)\ I_2 \tag{11}$$

Expanding Eq.(9) we can express the first order invariants of S_{ijkl} in terms of its components. In 2-space, i.e., for $i,j,k,\ldots = 1,2$, they are:

$$I_1 = S_{1111} + S_{2222} + 2S_{1122} \tag{12}$$

$$I'_1 = S_{1111} + S_{2222} + 2S_{1212} \tag{13}$$

$$I_2 = -4S_{1122} + 4S_{1212} \tag{14}$$

It is evident from Eq.(12),(13), and (14) that only two I's are independent. This is consistent with the induction that there are two first order invariants in 2-space for a 4th-order tensor. We arbitrarily assign I_1 and I_2 to be the first order invariants to be consistent with underived definitions given by Hearmon[1]. Also, to be consistent with the definition in Ref. [1], Eqs. (11 and 14) have been multiplied by-2.

The second order invariants which are hitherto not known can be arrived at in similar manners. By taking, what is in effect, the dot products of the cross products of the vector components of the 2nd-order tensor components $\left(D_{kl}, D'_{jl}, T_{nm}\right)$ of the 4th-order tensor $\left(S_{ijkl}\right)$, the following 2nd-order invariants can be arrived at.

$$\psi_1 = \delta_{pq}\ \varepsilon_{pik}\ \varepsilon_{qjl}\ T_{ij}\ T_{kl} = \left(\alpha_1 - 2\alpha_2 + \alpha_4\right) - \left(2\ \alpha_5 - \alpha_3\right) = 0 \quad (15)$$

$$\psi_2 = \delta_{pq}\ \varepsilon_{pik}\ \varepsilon_{qjl}\ D_{ij}\ D_{kl} = \alpha_1 - \alpha_6 \quad (16)$$

$$\psi_3 = \delta_{pq}\ \varepsilon_{pik}\ \varepsilon_{qjl}\ D'_{ij}\ D'_{kl} = \alpha_4 - \alpha_7 \quad (17)$$

$$\psi_4 = \delta_{pq}\ \varepsilon_{pik}\ \varepsilon_{qjl}\ D_{ij}\ D'_{kl} = \alpha_2 - \alpha_8 \quad (18)$$

$$\psi_5 = \delta_{pq}\ \varepsilon_{pik}\ \varepsilon_{qjl}\ D_{ij}\ T_{kl} = 2\left(\alpha_6 - \alpha_8\right) \quad (19)$$

$$\psi_6 = \delta_{pq}\ \varepsilon_{pik}\ \varepsilon_{qjl}\ D'_{ij}\ T_{kl} = 2\left(\alpha_8 - \alpha_7\right) \quad (20)$$

ψ_i can be expressed in terms of S_{ijkl} by using Eqs.(2,3 and 7). For example:

$$\psi_1 = \delta_{pq}\ \varepsilon_{pik}\ \varepsilon_{qjl}\ \varepsilon_{img}\ \varepsilon_{jnh}\ S_{mngh}\ \varepsilon_{ksu}\ \varepsilon_{ltv}\ S_{stuv} \quad (15')$$

$$\psi_2 = \delta_{pq}\ \varepsilon_{pik}\ \varepsilon_{qjl}\ \delta_{mn}\ S_{mnij}\ \delta_{uv}\ S_{uvkl} \quad (16')$$

For conciseness, the remaining of ψ_i are not expanded. The α_i expressed in terms of components of S_{ijkl} in 2-space are:

$$\alpha_1 = S_{iijj}\, S_{kkll} = S_{1111}^2 + 4S_{1111}S_{1122} + 2S_{1111}S_{2222} + 4S_{1122}^2 + 4S_{1122}S_{2222} + S_{2222}^2 \qquad (21)$$

$$\alpha_2 = S_{iijj}\, S_{klkl} = S_{1111}^2 + 2S_{1111}S_{1122} + 2S_{1111}S_{2222} + 2S_{1111}S_{1212} + 4S_{1122}S_{1212} + 2S_{1122}S_{2222} + 2S_{2222}S_{1212} + S_{2222}^2 \qquad (22)$$

$$\alpha_3 = S_{ijkl}\, S_{ikjl} = S_{1111}^2 + 4S_{1112}^2 + 4S_{1122}S_{1212} + 4S_{1222}^2 + 2S_{1212}^2 + S_{2222}^2 \qquad (23)$$

$$\alpha_4 = S_{ijij}\, S_{klkl} = S_{1111}^2 + 4S_{1111}S_{1212} + 2S_{1111}S_{2222} + 4S_{1212}^2 + 4S_{1212}S_{2222} + S_{2222}^2 \qquad (24)$$

$$\alpha_5 = S_{ijkl}\, S_{ijkl} = S_{1111}^2 + 4S_{1112}^2 + 2S_{1122}^2 + 4S_{1212}^2 + 4S_{1222}^2 + S_{2222}^2 \qquad (25)$$

$$\alpha_6 = S_{iikl}\, S_{jjkl} = S_{1111}^2 + 2S_{1111}^2S_{1122} + 2S_{1122}^2 + 2S_{1112}^2 + 4S_{2212}S_{1112} + 2S_{2212}^2 + 2S_{1122}S_{2222} + S_{2222}^2 \qquad (26)$$

$$\alpha_7 = S_{ikil}\, S_{jkjl} = S_{1111}^2 + 2S_{1111}S_{1212} + 2S_{1112}^2 + 4S_{1112}S_{2212} + 2S_{1222}^2 + 2S_{1212}S_{2222} + 2S_{1212}^2 + S_{2222}^2 \qquad (27)$$

$$\alpha_8 = S_{iijk}\, S_{ljlk} = S_{1111}^2 + S_{1111}\, S_{1212} + S_{1111}S_{1122} + 2S_{1112}^2 + 2S_{1122}S_{1212} + 4S_{1112}S_{2212} + 2S_{1222}^2 + S_{1122}S_{2222} + S_{1212}S_{2222} + S_{2222}^2 \qquad (28)$$

It can be shown that both ψ_i and α_i are invariant to coordinate transformations. Just as in the first order scalar contractions, not all of the 2nd-order scalar contractions, ψ_i or α_i, are independent. Gauss-Jordan reduction revealed that the system of equations Eq. (21) to (28) is of defect three. The three residual equations are:

$$\alpha_1 - \alpha_2 - 2\alpha_6 + 2\alpha_8 = 0 \tag{29}$$

$$\alpha_2 - \alpha_4 + 2\alpha_7 - 2\alpha_8 = 0 \tag{30}$$

$$\alpha_3 - \alpha_5 + \alpha_6 + \alpha_7 - 2\alpha_8 = 0 \tag{31}$$

In addition, there exist three relations between α's and the first invariants I_1, I_2 as can be shown from Eqs. (12), (14), (21), (23), (24) and (25).

$$I_1^2 = \alpha_1 \tag{32}$$

$$I_2^2 = 8\left(\alpha_5 - \alpha_3\right) \tag{33}$$

$$\left(I_1 + (1/2)I_2\right)^2 = \alpha_4 \tag{34}$$

Equations (21) through (34) indicate that out of the eight α's only two α's are independent. This is again consistent with the mathematical induction that there are two 2nd-order invariants for a 4th-order tensor in 2-space. Any two α's can be arbitrarily assigned as 2nd-order invariants. We chuse α_3 and α_7 and define the following combinations as 2nd-order invariants.

$$II_1 = 2\alpha_7 - \left(I_1 + (1/2)I_2\right)^2 \tag{35}$$

$$II_2 = 8\left(\alpha_3 - \alpha_7\right) + \left(I_1 + I_2\right)^2 \tag{36}$$

We will show in the following section that these particular combinations lend themselves to a convenient geometric interpretation akin to the Mohr's circle for 2nd-order tensors. Summarizing, we have derived 1st and 2nd-order scalar contractions for a 4th-order tensor. We also show that there are two independent 1st-order invariants and two independent 2nd-order invariants for a 4th-order tensor in 2-space. They can be expressed in terms of the components of the 4th-order tensor components as:

$$I_1 = S_{1111} + S_{2222} + 2S_{1122} \tag{37}$$

$$I_2 = -4S_{1122} + 4S_{1212} \tag{38}$$

$$II_1 = \left(S_{1111} - S_{2222}\right)^2 + 4\left(S_{1112} + S_{2212}\right)^2 \tag{39}$$

$$II_2 = \left(S_{1111} + S_{2222} - 2S_{1122} - 4S_{1212}\right)^2 + 16\left(S_{1112} - S_{2212}\right)^2 \tag{40}$$

In contracted notation*, the equivalent expressions are:

$$I_1 = S_{11} + S_{22} + 2S_{12} \tag{37'}$$

$$I_2 = -4S_{12} + S_{66} \tag{38'}$$

$$II_1 = \left(S_{11} - S_{22}\right)^2 + \left(S_{16} + S_{26}\right)^2 \tag{39'}$$

$$II_2 = \left(S_{11} + S_{22} - 2S_{12} - S_{66}\right)^2 + 4\left(S_{16} - S_{26}\right)^2 \tag{40'}$$

---

*The contracted notation follows the common practice of representing a 4-th order tensor with two indexes where $S_{1111} \equiv S_{11}$, $S_{1122} \equiv S_{12}$, $2S_{1112} \equiv S_{16}$, $2S_{2212} \equiv S_{26}$, $4S_{1212} \equiv S_{66}$. For more detail see Ref. [1].

Geometric Interpretation of 4th-Order Tensor Invariants

Because of the large number of components involved, the geometric interpretation of tensors of order higher than two is a formidable task. Even in the plane case, the geometric representation for a 2nd-order tensor is equivalent to represent in two dimensions a system of three quadratic trigonometric functions. Mohr's circle provides such a representation. It contributed to better visualization of a 2nd-order tensor and provided greater operational convenience for engineering problems in the analysis of stress and strain as well as for problems in rigid body dynamics. Similarly, the representation of a 4th-order tensor is equivalent to representing a system of 4th-order trigonometric functions. A geometric representation similar to that of the Mohr's circle has been suggested by P. Mast [2]. Utilizing the invariants derived in the previous section Eqs. (37), (38), (39) and (40), this can be generalized to geometrically represent the transformation of a 4th-order tensor from any arbitrary non-principal direction.

From the definition of tensors, for rotation about the x_3 axis, the components of a 4th-order tensor S_{ijkl} at an arbitrary orientation θ with respect to a material coordinate* can be transformed to the components S'_{ijkl} at another orientation θ' through the relation [1]:

---

*Without loss of generality, we may consider the material coordinate coincides with the principal direction.

$$\begin{bmatrix} S'_{1111} \\ S'_{1122} \\ 2S'_{1112} \\ S'_{2222} \\ 2S'_{2212} \\ 4S'_{1212} \end{bmatrix} = \begin{bmatrix} m^4 & 2m^2n^2 & 2m^3n & n^4 & 2mn^3 & m^2n^2 \\ m^2n^2 & m^4+n^4 & mn^3-m^3n & m^2n^2 & m^3n-mn^3 & -m^2n^2 \\ -2m^3n & 2(m^3n-mn^3) & m^4-3m^2n^2 & 2mn^3 & 3m^2n^2-n^4 & m^3n-mn^3 \\ n^4 & 2m^2n^2 & -2mn^3 & m^4 & -2m^3n & m^2n^2 \\ -2mn^3 & 2(mn^3-m^3n) & 3m^2n^2-n^4 & 2m^3n & m^4-3m^2n^2 & mn^3-m^3n \\ 4m^2n^2 & -8m^2n^2 & 4(mn^3-m^3n) & 4m^2n^2 & 4(m^3n-mn^3) & (m^2-n^2)^2 \end{bmatrix} \begin{bmatrix} S_{1111} \\ S_{1122} \\ 2S_{1112} \\ S_{2222} \\ 2S_{2212} \\ 4S_{1212} \end{bmatrix} \qquad (41)$$

where m and n are directional cosine and sine.

The graphical representation of this 4th-order tensor transformation can be observed by rewriting Eq. 41 in multiple angle representation as described by Tsai and Pagano [3] in the following form:

$$\begin{bmatrix} S'_{1111} \\ S'_{1122} \\ 2S'_{1112} \\ S'_{2222} \\ 2S'_{2212} \\ 4S'_{1212} \end{bmatrix} = \begin{bmatrix} U_1 & -U_2 & U_6 & -U_3 & U_7 \\ -U_4 & 0 & 0 & U_3 & -U_7 \\ 0 & U_6 & U_2 & 2U_7 & 2U_3 \\ U_1 & U_2 & -U_6 & U_3 & 2U_7 \\ 0 & U_6 & U_2 & -2U_7 & -2U_3 \\ 4U_5 & 0 & 0 & 4U_3 & -4U_7 \end{bmatrix} \begin{bmatrix} 1 \\ \cos 2\phi \\ \sin 2\phi \\ \cos 4\phi \\ \sin 4\phi \\ 0 \end{bmatrix} \qquad (42)$$

where

$$\phi = \theta - \theta'$$

$$U_1 = 1/8\left(3I_1 + I_2\right)$$

$$U_2 = 1/2\left(S_{2222} - S_{1111}\right)$$

$$U_3 = 1/8\left(4S_{1212} + 2S_{1122} - S_{1111} - S_{2222}\right) \qquad (43)$$

$$U_4 = 1/8\left(I_2 - I_1\right)$$

$$U_5 = 1/8\left(I_2 + I_1\right)$$

$$U_6 = \left(S_{1112} + S_{2212}\right)$$

$$U_7 = 1/2\left(S_{1112} - S_{2212}\right)$$

The geometric interpretation of this tensor transformation can be recognized after some rearrangements. Take the first of Eq. (42) for example,

$$S'_{1111} = U_1 - U_2 \cos 2\phi + U_6 \sin 2\phi - U_3 \cos 4\phi + U_7 \sin 4\phi \qquad (44)$$

Consider the terms containing 2ϕ and define:

$$U'_2 \equiv U_2 \cos 2\phi - U_6 \sin 2\phi \qquad (45)$$

If U_2 and U_6 are the sides of a right-angled triangle with a hypothenuse R_1, as shown in Fig. 1, then

$$U_2 = R_1 \cos 2\theta$$

$$U_6 = R_1 \sin 2\theta \qquad (46)$$

$$R_1 = U_2 \cos 2\theta + U_6 \sin 2\theta$$

Substitute Eq. (46) into (45), we obtain

$$U_2' = (U_2 \cos 2\theta + U_6 \sin 2\theta)\cos(2\theta + 2\phi) \tag{47}$$
$$= R_1 \cos 2\theta'$$

Similarly, we can express terms containing 4ϕ as:

$$U_3 \cos 4\phi - U_7 \sin 4\phi = R_2 \cos 4\theta' \tag{48}$$

where

$$R_2 = U_3 \cos 4\phi + U_7 \sin 4\phi$$

Making use of Eq. (47) and (48), Eq. (44) simplifies into the form:

$$S_{1111}' = U_1 - R_1 \cos 2\theta' - R_2 \cos 4\theta' \tag{49}$$

As shown in Fig. 2, Eq. (49) can be represented by the horizontal projected distance between the radius vectors of two circles separated by distance U_1 between centers and with radii R_1 and R_2 respectively. The radius vector for R_1 rotates at $2\theta'$ and that for R_2 rotates at $4\theta'$. Both the original S_{1111} at θ and the transformed S_{1111}' at θ' are shown in Fig. 2. It can also be easily shown that R_1 and R_2 are the roots of the 2nd invariants [Eq. (39), (40)].

$$\begin{aligned} R_1 &= \left(U_2^2 + U_6^2\right)^{1/2} = 1/2\left(II_1\right)^{1/2} \\ R_2 &= \left(U_3^2 + U_7^2\right)^{1/2} = 1/8\left(II_2\right)^{1/2} \end{aligned} \tag{50}$$

Carrying out similar rearrangements, Eq. (42) can be written as:

$$\begin{bmatrix} S'_{1111} \\ S'_{1122} \\ S'_{1112} \\ S'_{2222} \\ S'_{2212} \\ S'_{1212} \end{bmatrix} = \begin{bmatrix} U_1 & -R_1 & 0 & -R_2 & 0 \\ -U_4 & 0 & 0 & R_2 & 0 \\ 0 & 0 & 1/2R_1 & 0 & R_2 \\ U_1 & R_1 & 0 & -R_2 & 0 \\ 0 & 0 & 1/2R_1 & 0 & -R_2 \\ U_5 & 0 & 0 & -R_2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \cos 2\theta' \\ \sin 2\theta' \\ \cos 4\theta' \\ \sin 4\theta' \\ 0 \end{bmatrix} \tag{51}$$

The components of the 4th-order tensor at orientation θ (which is equivalent to $\phi = 0$ in Eq. 51) can be represented geometrically in Fig. 3. In the construction of this representation, it is assumed that:

$$\begin{aligned} &U_2 > 0 \text{ or } S_{2222} > S_{1111} \\ &U_3 > 0 \text{ or } \left(4S_{1212} + 2S_{1122}\right) > \left(S_{1111} + S_{2222}\right) \\ &U_4 > 0 \text{ or } 4S_{1212} > \left(S_{1111} + S_{2222} + 6S_{1122}\right) \end{aligned} \tag{52}$$

In addition, just as in the Mohr's circle, certain sign conventions must be followed. A positive rotation ϕ (counter-clockwise) in the material plane is equivalent to a positive rotation of 2ϕ (counter-clockwise) of radius vector R_1 and a positive rotation 4ϕ (also counter-clockwise) of the radius vector R_2. The signs of the components S_{ijkl} are determined by the direction of the measurements. All measurements from left to right and those from bottom to top are positive. In order to avoid confusion, the directions of measurements as indicated by the arrows in Fig. 3 must be followed. For example, S_{1112} is obtained by the vertical distance from the tail of the diameter vector of circle R_2 to the head of diameter vector of circle$(1/2)R_1$. Note that when $\theta > \pi/2$, the measurement direction for S_{1112} is reversed and S_{1112} takes on negative values.

The above geometric representation can be constructed from any given set of S_{ijkl}. The orientation of the principal direction need not be known. The invariants I_1, I_2, II_1, II_2 can be computed from Eqs. (37,38,39 and 40). From Eqs.(43 and 50), U_1, U_4, U_5, R_1 and R_2 can be computed. These values can be used to locate the centers of the circles, reference lines A and B and the circles R_1,$(1/2)R_1$,and R_2 as shown in Fig. 3. Either S_{1112} or S_{1212} can then be used to determine the orientation of the radius vector R_2. Thus the principal direction θ can be directly determined, which in turn determines the radius vector R_1. The components of the 4th-order tensor S'_{ijkl} at orientation ϕ from S_{ijkl} can be determined by rotating the R_1 by 2ϕ and R_2 by 4ϕ and the appropriate distances measured according to the suggested sign conventions.

The representation presented herein is but one of many available. It serves, primarily, to illustrate the essential features which exist in the geometrical representation of a 4th-order tensor. Detail variations will depend on personal preferences and specific application. For example, if the transformation of S_{ijkl} in contracted notation is desired, the third, fifth and sixth equations of Eq. (51) must be multiplied by

2, 2 and 4 respectively. It follows that in the geometric representation, the circle $(1/2)R_1$ can be removed and concentric circles $2R_2$ and $4R_2$ added to the cricle R_2. In addition, the location of reference lines A and B as well as the relative positions of the circles are arbitrary. If they are changed from the configuration presented here, appropriate sign convention has to be re-established.

It is also worthwhile to note the analogy between the Mohr's circle transformation for 2nd-order tensors and the representation for 4th-order tensors. In Mohr's circle, the first order invariant determines the location of the center of the circle, and the 2nd-order invariant determines the magnitude of the circle. In the 4th-order representation, the first order invariants determine the location of the centers of the circles and the reference lines A and B while the 2nd-order invariants determine the magnitudes of the circles. For a rotation ϕ in the physical plane, the second order transformation is represented by a rotation 2ϕ in the Mohr's circle while the 4th-order transformation is represented by rotations of 2ϕ and 4ϕ of the circles.

In practical application, this geometric representation is used to graphically transform the compliance matrix S_{ijkl} of fiberglass reinforced composites. It was found that upon the computation of the invariants and the construction of the circles, any S'_{ijkl} can be rapidly determined by laying out two vectors R_1 and R_2 at the desired orientations 2θ and 4θ. From constructions on 8"x 10"graph papers*,the accuracy attained is within 1% of the computer computed results using Eq.(41).

*Graphical construction can be simplified by normalizing Eq.(51) by R_1 or R_2 or U_1.

In cases where principal direction is to be found, it can be determined graphically by the intercept of S'_{1212} to the circle R_2, and this is found to be much faster than analytically setting S'_{1112} or S'_{2212} to zero in Eq. (41) and solve for θ.

Applications

We have derived the first and second order invariants for 4th-order tensors and have shown that in 2-space there exists two independent first order invariants and two independent 2nd-order invariants. By expressing the 4th-order tensor transformation in multiple angle representation and making use of the invariants, a geometric representation analogous to the Mohr's circle can be constructed.

The invariants and graphical representation presented here are applicable to 4th-order tensors encountered in the physics and mechanics of solids and fluids. They are also applicable for certain engineering constants for composite materials which transform in the form of Eq. (1). Thus, similar invariants and geometric representations can be derived for the $\hat{A}$, $\tilde{B}$, and $\tilde{D}$ matrix for laminate plates [4]. Familiar applications of the invariants of 2nd-order tensors and Mohr's circle suggest some natural applications of their counterparts for 4th-order tensors. For example, the principal values and principal directions of the stiffness tensor for crystals and composite materials can be conveniently obtained from the invariants and the graphical representation. In view of the contribution of stress invariants to the formulation of yield criterion for isotropic materials, the 4th-order tensor invariants may be useful in the characterization of the flow and fracture of anisotropic solids. Finally, the newly derived second order invariants may be considered as additional intrinsic material properties and may be used to simplify optimization of the physical properties of laminated composites as suggested in Ref. [3].

Appendix

Invariants of 4th-order tensor in 3-space:

The derivation of the invariants of a 4th-order tensor in 3-space for a rotation of coordinate around the X_3-axis is similar to that in 2-space. Because $X_3 = X_3'$ in the transformation, the components of S_{ijkl} which contain the 3 component transform as tensors of orders less than four; i.e., the components

S_{1123}, S_{1113}, S_{2223}
S_{2213}, S_{2312}, S_{1312} Transforms as 3rd-order tensor (A1)

$$\left.\begin{matrix} S_{1313}, S_{2323}, S_{1323} \\ S_{1133}, S_{2233}, S_{3312} \end{matrix}\right\} \quad \text{Transform as 2nd-order tensor} \qquad \text{(A2)}$$

S_{3323}, S_{3313} Transform as 1st-order tensor (A3)

Thus, in addition to the invariants derived (Eq. 37 to 40), there exist invariants associated with the tensor components in Eq.(A1 to A3). For the components in Eq.(A1), it can be easily shown that there exists no 1st-order invariants. The 2nd-order invariants are:

$$II_3 = \left(S_{1123} - S_{2223} + 2S_{1312}\right)^2 + \left(S_{1113} - S_{2213} - 2S_{2312}\right)^2 \qquad \text{(A4)}$$

$$II_4 = \left(3S_{1123} + S_{2223} - 2S_{1312}\right)^2 + \left(S_{1113} + 3S_{2213} - 2S_{2312}\right)^2 \qquad \text{(A5)}$$

$$II_5 = \left(3S_{1113} + S_{2213} + 2S_{2312}\right)^2 + \left(S_{1123} + 3S_{2223} + 2S_{1312}\right)^2 \qquad \text{(A6)}$$

$$II_6 = \left(S_{1113} - S_{2213} + 2S_{2312}\right)^2 + \left(S_{1123} - S_{2223} - 2S_{1312}\right)^2 \qquad \text{(A7)}$$

For the components in Eq. (A2), the first and second order invariants can be obtained from well known results of 2nd-order tensor transformation. The first order invariants are

$$I_3 = 4S_{2323} + 4S_{1313} \tag{A8}$$

$$I_4 = S_{2233} + S_{1133} \tag{A9}$$

The second order invariants are

$$II_7 = \left(S_{1133} - S_{2233}\right)^2 + \left(2S_{3312}\right)^2 \tag{A10}$$

$$II_8 = \left(S_{2323} - S_{1313}\right)^2 + \left(4S_{2312}\right)^2 \tag{A11}$$

Finally, for the components in Eq.(A3), no first order invariants exist and the second order invariant is the magnitude of the vector

$$II_9 = \left(S_{3323}\right)^2 + \left(S_{3313}\right)^2 \tag{A12}$$

Recapitulating, for a 4th-order tensor S_{ijkl} in 3-space in a rotation around the X_3-axis, the 1st-order invariants are: I_1(Eq.37), I_2(Eq.38), I_3(Eq.A8), I_4(Eq.A9),
and the 2nd order invariants are II_1(Eq.38), II_2(Eq.39), II_3(Eq.A4), II_4(Eq.A5), II_5(Eq.A6), II_6(Eq.A7), II_7(Eq.A10), II_8(Eq.A11), II_9(Eq.A12).

The geometric representations of the transformation of the components in Eq.(A2) and Eq.(A3) are that of the conventional Mohr's circle and that of the vector circle respectively. The geometric representation of the components in Eq.(A1) can be seen by expressing the transformation in multiple angles using the relationships

$$\begin{aligned} \sin^3\theta &= 1/4(3\sin\theta - \sin 3\theta) \\ \cos\theta\sin^2\theta &= 1/4(\cos\theta - \cos 3\theta) \\ \cos^2\theta\sin\theta &= 1/4(\sin\theta + \sin 3\theta) \\ \cos^3\theta &= 1/4(3\cos\theta + \cos 3\theta) \end{aligned} \tag{A13}$$

$$\begin{bmatrix} S'_{1123} \\ S'_{1113} \\ S'_{2223} \\ S'_{2213} \\ S'_{2312} \\ S'_{1312} \end{bmatrix} = \begin{bmatrix} R_4 & 0 & R_3 & 0 \\ 0 & R_5 & 0 & R_3 \\ R_5 & 0 & -R_3 & 0 \\ 0 & R_4 & 0 & -R_3 \\ 0 & -R_6 & 0 & -R_3 \\ -R_6 & 0 & R_3 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \\ \cos 3\theta \\ \sin 3\theta \end{bmatrix} \qquad \text{(A14)}$$

where

$$R_3 = 1/2\left(II_3\right)^{1/2},\ R_4 = 1/2\left(II_4\right)^{1/2},\ R_5 = 1/2\left(II_5\right)^{1/2},$$
$$R_6 = 1/4\left(II_6\right)^{1/2}$$

The geometric representation of equation (A14) is similar to the one constructed for Eq.(51) and is shown in Fig. 4. In this construction, II_3, II_4, II_5, II_6 are assumed to be positive. It can be noted that in the principal direction ($\theta = 0$), $S_{1113} = S_{2213} = S_{2312} = 0$ indicating that the material is symmetric to the 2-3 plane, i.e.,a mono-clinic material with X_1 as the principal axis. It is important to note that since the transformation of S_{ijkl} in 2-space (Eq. 41) is not coupled to the transformations of the S_{ijkl} components in 3-space (Eqs.A1, A2 and A3), the invariants Eqs. (37-40) and geometric representation (Fig. 3) remain the same for 3-space. Previous remarks on the applications of the invariants and geometric representation for 2-space are also applicable for their counterparts in 3-space.

References

1. R.F.S. Hearmon, An Introduction to Applied Anisotropic Elasticity, Oxford University Press, 1961.

2. P. Mast, private communication, Naval Research Laboratory.

3. S.W. Tsai and N.J. Pagano, "Invariant Properties of Composite Materials," Composite Materials Workshop, Technomic Publishing Co., 1968.

4. J. E. Ashton, J. C. Halpin and P. H. Petit, Primer on Composite Materials: Analysis, Technomic Publishing Co., 1969.

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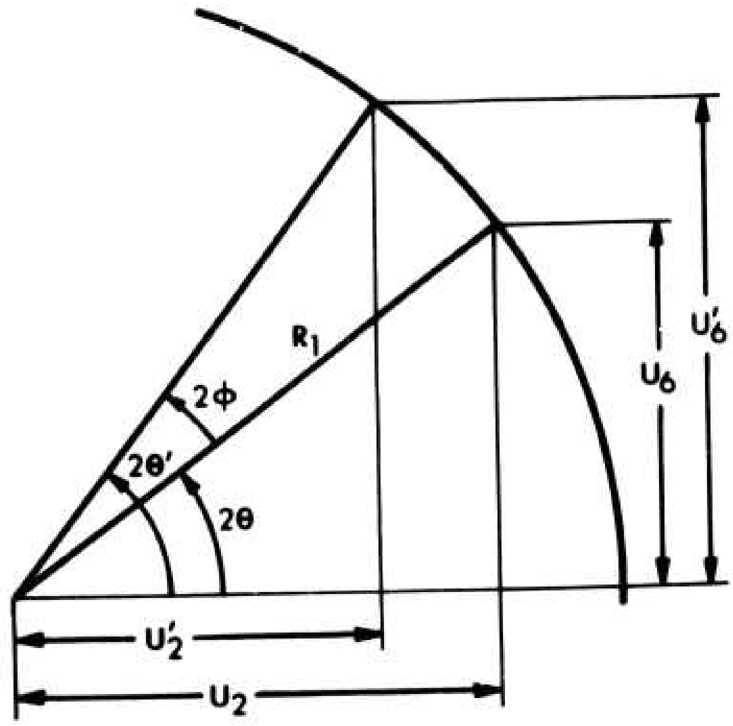

FIGURE 1

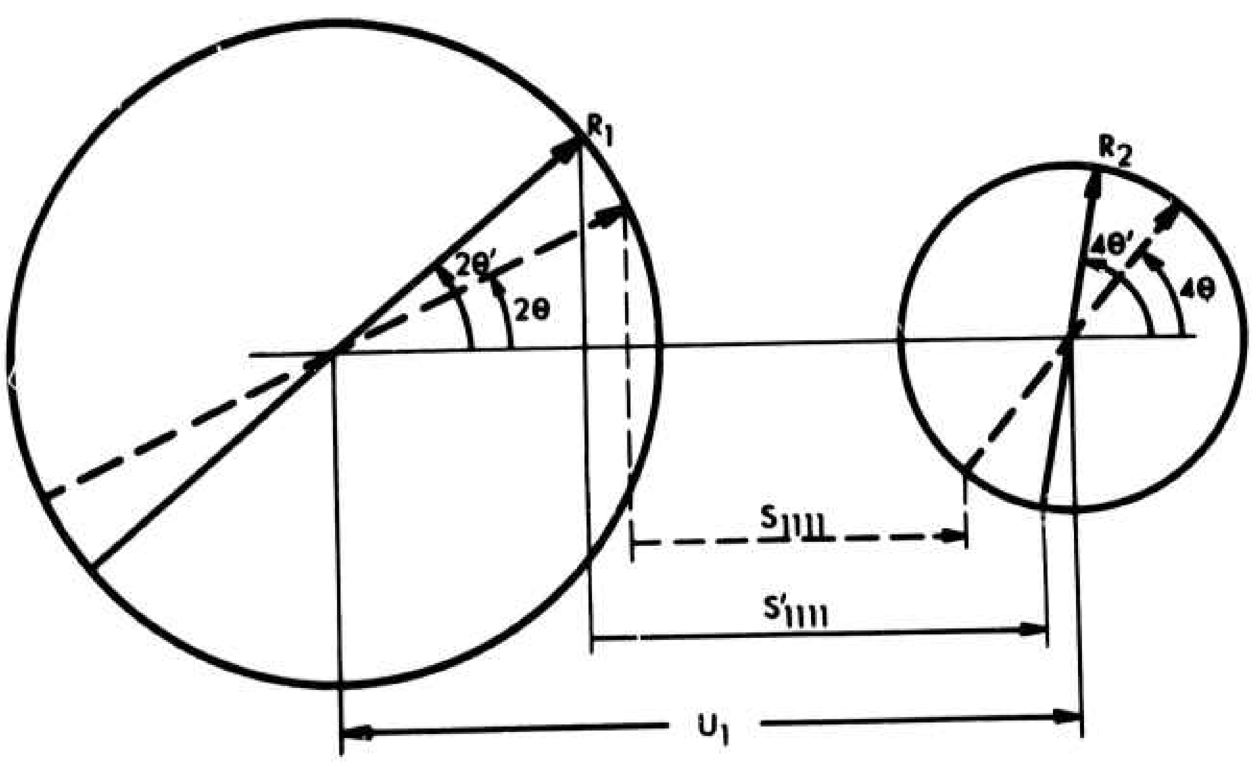

FIGURE 2

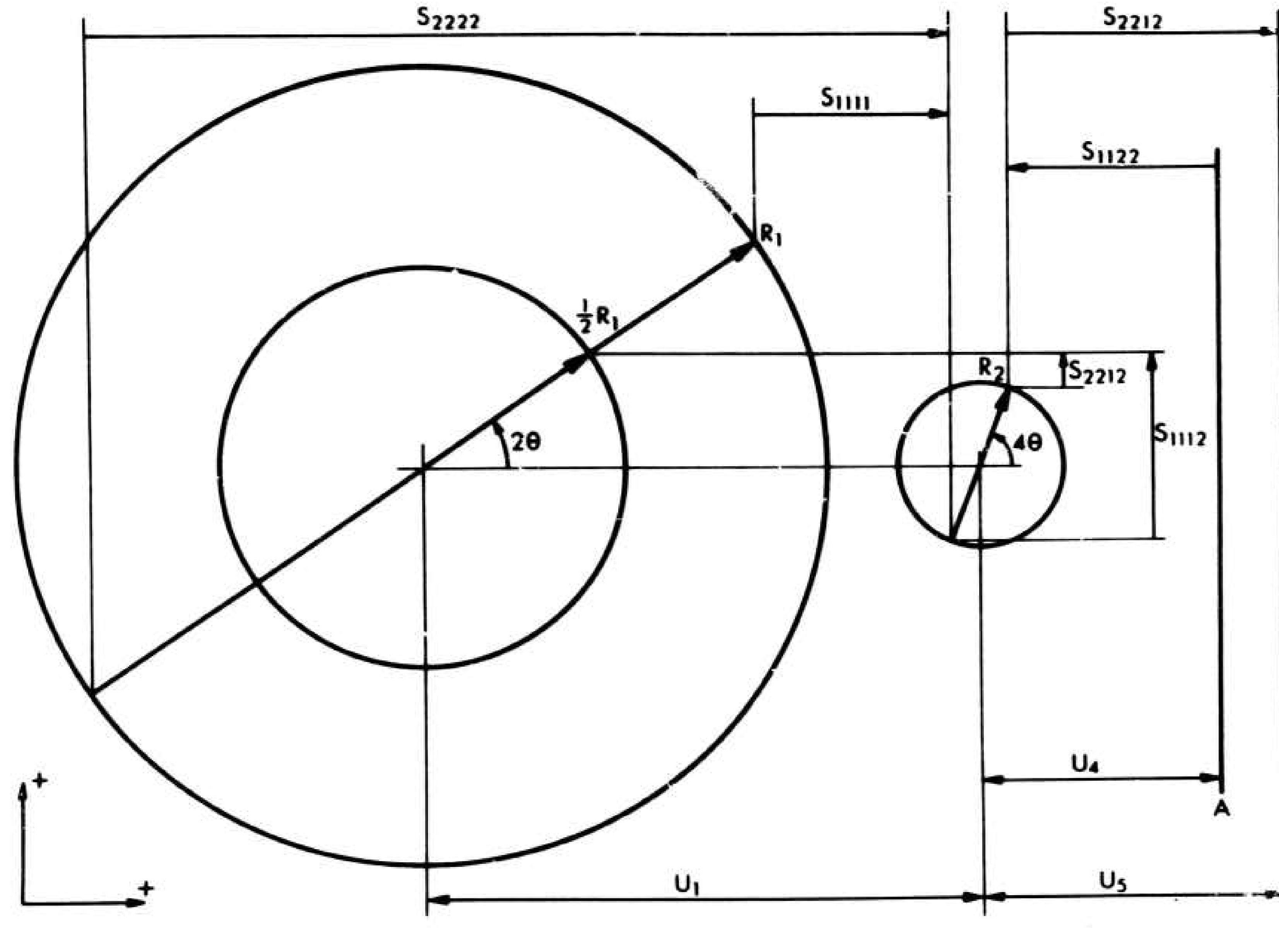

FIGURE 3. TRANSFORMATION OF FOURTH ORDER TENSOR

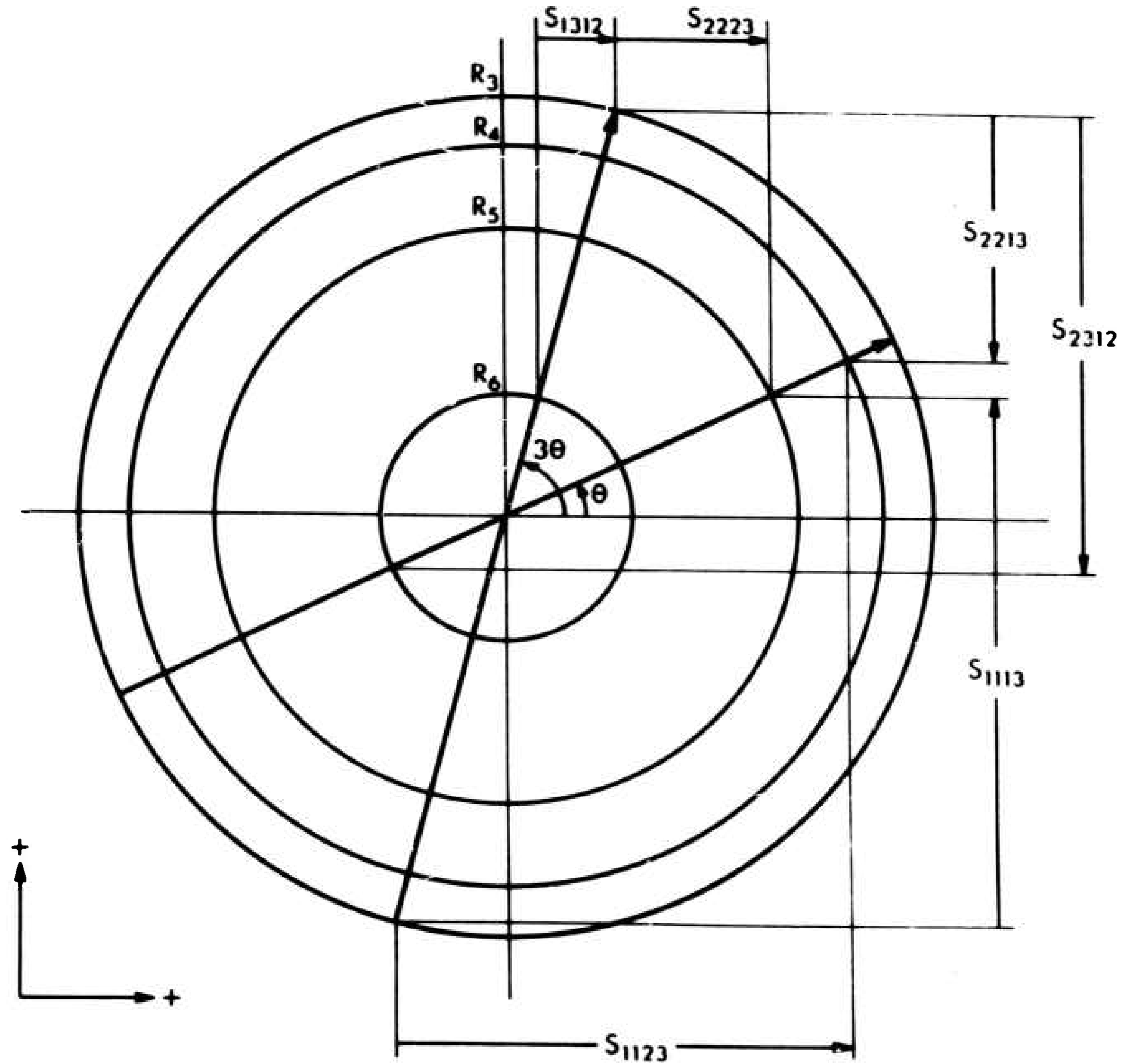

FIGURE 4. TRANSFORMATION OF THIRD ORDER TENSOR

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13. ABSTRACT

First and second order invariants of 4th-order tensor are derived. Geometric representations analogous to Mohr's circle are presented to aid the visualization and operations of 4th-order tensors such as transformation, determination of principal direction and optimization procedures. Possible applications of the invariants are suggested.

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